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# Hyperbolic elasticity of dissipative media and its wave propagation modes

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**Abstract.** A phenomenological theory of dissipative elastic solids whose equations form a hyperbolic system is proposed. The Müller–Israel non-stationary transport equations for dissipative fluxes containing new cross-effect terms, as required by compatibility with irreversible thermodynamics, have been adopted. As opposed to usual conventional theories, which are parabolic and predict instantaneous propagation of wavefronts, the theory formulated, formed of 14 partial differential equations, all of 17th order, is hyperbolic and predicts, for all existing propagation modes, finite front speeds. The complete system of propagation modes is determined from the linearized equations. There are four mutually distinct non-trivial propagation modes, two for longitudinal waves and two for transverse waves. The slow transverse mode (quasithermal) was predicted for the first time, while the remaining modes were improved.

## 1. Introduction

The predictions of the theory of elasticity for wave propagation in solids, if dissipation is neglected, have been very well verified by experiments, at least for normally behaved solids. This is one of the most important reasons mitigating for the general acceptance of such a dynamical theory of non-dissipative elastic solids. However, theories including the effects of dissipation are much less satisfactory. The most common weakness of existing conventional theories (e.g. viscoelasticity in a Voigt solid) is that the resulting system of partial differential equations is not hyperbolic, which means that an infinite front speed for propagating waves is predicted. Thus the physical causality between wave source and signal reception is violated. The defect can be traced to the ommission of relaxation terms (i.e. non-stationary terms) in the transport equations for dissipative fluxes.

Recent developments in kinetic theory (Grad 1949) have clarified the form of the transport equations for both the heat flux and viscosity tensor†. It is not important that this was done only for a dilute gas because it gave a firm basis for a revision of the transport equations. (Note that Maxwell (1890) was motivated, by an analysis of the nature of viscosity in gases, to propose his relaxed stress–strain formula.) The idea of a heat equation with relaxed heat flux, advocated by Cattaneo (1948) and others, gradually gained more acceptance. The Fourier equation including relaxation was soon

† The idea of equivalence of the Boltzmann equation with an infinite system of moment equations (which can gradually be made explicit) can be traced back to Maxwell (see Ikenberry and Truesdell 1956).

proposed and used also in elasticity (see e.g. Eringen 1960, Popov 1967, Lord and Schulman 1967, Achenbach 1968, Norwood and Warren 1969, McCarthy 1970a, b, Kaliski 1965 and others). The most visible defect of Voigt's transport equation for viscoelastic media was also amended by the theory of relaxation or by the use of a combined Maxwell-Voigt stress-strain law. But all this still has one weakness: the relaxed transport equations are considered as mutually independent† and compatability of their corrected form with thermodynamics was ignored. We notice that only dissipative fluxes are *relaxed*, not the reversible quantities and the role of the driving force and its response must not be interchanged as relaxation terms $\tau\, d/dt$ are not invariant against a substitution $t \rightarrow -t$. Thermodynamics of the Maxwell solid, where the total stress is relaxed, has not yet been constructed (Eringen 1967a, b). The same also applies to the Voigt-Maxwell solid. For the purely reversible quantities such as $\sigma_{kl}$ in equation (2.5), the effect of finite velocity propagation is still secured by the inertial term (d'Alembert's force) in the equation of motion.

If, for example, a traditional stationary Gibbs equation is used, together with relaxed equations (2.9)–(2.11), one deduces that the derivative of fluxes also appears in entropy production $\sigma$ (Kranyš 1967, equation (1.32), McCarthy 1970a, b) which is incorrect. In reality, according to non-equilibrium thermodynamics, retaining systematically all second-order terms in the entropy balance equation enables the more accurate form of the transport equations for dissipative fluxes to be predicted (Müller 1967). The purpose of this paper is firstly, to propose a theory of an isotropic elastic dissipative continuum which is hyperbolic (requiring the relaxed transport equation), and at the same time, strictly consistent with the principles of irreversible thermodynamics. As we will see, this requires the inclusion of some cross-effect coupling terms in the transport equations and a modified form of some thermodynamic equations such as, for example, the Gibbs equation. Secondly, we deduce all the possible propagation modes (in an unbounded medium) according to such a theory and compare them with some which are well known.

## 2. Formulation of phenomenological theory of dissipative solid

Internal friction (in a wide sense) in solids may be produced by several different mechanisms, and although ultimately these all result in mechanical energy being transformed into heat, two different dissipative processes are involved. These two processes are roughly the counterparts of viscosity losses and thermal conduction losses in the transmission of sound waves through fluids. For fluids the dissipative effects are due to viscosity and thermal conduction and these effects can be investigated analytically (based on a phenomenological theory) in a quite satisfactory way if suitable transport equations are used. There is no doubt that this approach owes a great deal to the merit of kinetic theory, which clarified the structure of phenomenological theory valid for fluids in general. In spite of the fact that the behaviour is found to be more complex in solids, varying considerably with the nature of the solid, the phenomenological theories considering dissipation in terms of internal viscosity and heat conduction are generally used. Among them we can distinguish older theories like those which

† If only some of the retained transport equations are used in the non-stationary form while others are left in a stationary form the whole system is not hyperbolic. Also disregarding heat flux is not to be recommended because it acts as the coupling between various kinds of energies.

use the Meyer–Kelvin–Voigt stress–strain relation together with Fourier's law to form a parabolic system, and recent theories which use an appropriate transport equation converted into the non-stationary form, by including relaxation terms, to form a hyperbolic system together with conservation equations. In the process for including non-stationary relaxation terms in transport equations, one may eventually ask whether the additional new terms are of similar order of magnitude, what their form is, and one may show whether the new terms are compatible with non-equilibrium thermodynamics.

Indeed, thermodynamical considerations can be used to predict the more general form of transport equations for viscosity and heat conduction rather than the form of constitutive equations used in present theories.

The purpose of this paper is to propose a phenomenological theory for dissipative solids which is hyperbolic and, at the same time, strictly consistent with non-stationary thermodynamics.

In order to establish such a theory let us make the following postulates.

(i) First we assume (as in Voigt 1892) that the total stress components $\overset{*}{\tau}_{kl}$ in a solid can be expressed as a sum of the reversible or recoverable part of stress $\sigma_{kl}$ and the irreversible or dissipative part of stress $\mathcal{T}_{kl}$ (see also e.g. Truesdell and Toupin 1960 or Eringen 1967a):

$$\overset{*}{\tau}_{kl} = \sigma_{kl} + \mathcal{T}_{kl}, \qquad \overset{*}{\tau}_{kl} = \overset{*}{\tau}_{lk}, \qquad \mathcal{T}_{kl} = \mathcal{T}_{lk}, \qquad (k, l = 1, 2, 3)\dagger. \tag{2.1}$$

As a consequence of this postulate, $\sigma_{kl}$, as a fully reversible perfectly elastic stress, possesses the elastic potential $\psi'$ which is the reversible part of the Helmholtz free energy $\psi$. In the following, for the sake of clarity, we limit ourselves only to small, so called infinitesimal deformations $e_{kl}$ although such a limitation is not necessary for the development of our treatise. With this in mind, we can write

$$-\sigma_{kl} = \rho\left(\frac{\partial\psi'}{\partial e_{kl}}\right)_T, \qquad \psi' = \epsilon - \eta' T, \qquad \eta' = -\frac{\partial\psi'}{\partial T}. \tag{2.2}$$

Assuming

$$\epsilon = \epsilon(e_{kl}, T), \qquad \eta' = \eta'(e_{kl}, T) \tag{2.3}$$

and making use of the fact that $\mathrm{d}\epsilon$ and $\mathrm{d}\eta'$ are both exact differentials one can deduce the equation (see Chadwick 1964, equation (2.28)):

$$\rho\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \rho c_e \frac{\mathrm{d}T}{\mathrm{d}t} - \sigma_{kl}\frac{\mathrm{d}e_{kl}}{\mathrm{d}t} + T\frac{\partial\sigma_{kl}}{\partial T}\frac{\mathrm{d}e_{kl}}{\mathrm{d}t}; \qquad c_e = \left(\frac{\partial\epsilon}{\partial T}\right)_e = T\left(\frac{\partial\eta'}{\partial T}\right)_e, \tag{2.4}$$

$c_e$ being the specific heat at constant strain.

For isotropic, linearly elastic bodies (consistent with an infinitesimal deformation) one deduces from equations (2.2) and (2.3)

$$-\sigma_{kl} = A\delta_{kl} + \lambda\delta_{kl}e_{ss} + 2\mu e_{kl} - \beta(T - T_0)\delta_{kl}; \qquad \beta = (3\lambda + 2\mu)\alpha, \tag{2.5}$$

† In order that the signs of all terms in the conservation and transport equations (written with zero on the right-hand side) be the same (as it is in Müller's notation) we chose an opposite sign for $\overset{*}{\tau}_{kl}$ and $\sigma_{kl}$ than is commonly used.

where the conventional notation has been adopted, i.e.

$u_l(\boldsymbol{x}, t)$ is the displacement field, $\rho$ denotes mass density
$v_l = \mathrm{d}u_l/\mathrm{d}t \equiv \dot{u}_l$ is the displacement velocity field
$e_{kl} = \partial_{(k}u_{l)} = \frac{1}{2}(\partial_k u_l + \partial_l u_k)$ is the deformation tensor $(e_{kl} = \frac{1}{3}\delta_{kl}e_{ss} + \langle e_{kl}\rangle)$
$\langle \partial_k v_l\rangle = \partial_{(k}v_{l)} - \frac{1}{3}\delta_{kl}\,\partial_s v_s \equiv \langle \partial_k \dot{u}_l\rangle = \langle \dot{e}_{kl}\rangle$
$\langle \partial_k v_l\rangle = \partial_{(k}v_{l)} - \frac{1}{3}\delta_{kl}\,\partial_s v_s \equiv \langle \partial_k \dot{u}_l\rangle = \langle \dot{e}_{kl}\rangle$ denotes the deviator of deformation rate
$\alpha$ represents the coefficient of thermal expansion (2.6)
$T_0$ is the temperature of the reference natural state
$\lambda$ and $\mu$ are Lamé's constants
$A\delta_{kl}$ denotes remanent stress or initial stress in the natural state
$\epsilon$ is the specific internal energy.

Equation (2.5) represents Hooke's law for a thermally coupled elastic solid.

The tensor $\mathcal{T}_{kl}$, representing irreversible internal friction, which we will also write in the form

$$\mathcal{T}_{kl} = \tfrac{1}{3}\delta_{kl}\pi + \tau_{kl}, \qquad \mathcal{T}_{ss} = \pi, \qquad \tau_{ss} = 0 \tag{2.7†}$$

is conventionally connected with the tensor of deformation velocity by the proportionality relation, as in rigid body mechanics, where the friction force is usually considered to be proportional to the relative velocity of the two bodies in contact:

$$-\mathcal{T}_{kl} = \lambda'\delta_{kl}\dot{e}_{ss} + 2\mu'\dot{e}_{kl} \qquad \text{or} \qquad \begin{cases} \pi = -(3\lambda' + 2\mu')\dot{e}_{ss} \\ \tau_{kl} = -2\mu'\langle \dot{e}_{kl}\rangle \end{cases} \tag{2.8}$$

by equations (2.7) and (2.6) in analogy to Hooke's law. Here $\lambda'$ and $\mu'$ are moduli of viscosity which correspond to Lamé's constant. In viscoelasticity equations (2.8) together with (2.1) and (2.5) are used to describe a so called Voigt solid, while equations (2.5) are known as Newton's laws in fluid theory. Equations (2.8) describe the instantaneous dissipative stress response ($\pi$ and $\tau_{kl}$) arising from the strain velocities ($\dot{e}_{ss}$ and $\langle \dot{e}_{kl}\rangle$) which contradicts physical causality and must therefore be corrected. For this correction we propose the following.

(ii) We assume that transport equations for dissipative fluxes in an elastic continuum have the form:

$$\pi + \tau^0\dot{\pi} = -\lambda^0(\dot{e}_{ss} + 3N\,\partial_s q_s), \qquad \lambda^0 = 3\lambda' + 2\mu'; \tag{2.9†}$$

$$\tau_{kl} + \bar{\tau}\dot{\tau}_{kl} = -\bar{\lambda}(\langle \dot{e}_{kl}\rangle + M\langle \partial_k q_l\rangle), \qquad \bar{\lambda} = 2\mu'; \tag{2.10}$$

$$q_l + \tau\dot{q}_l = -\kappa(\partial_l T + NT\,\partial_l \pi + MT\,\partial_s \tau_{ls}); \tag{2.11}$$

where five new coefficients (in general dependent on $T$ and $\rho$), in comparison to the stationary theory, are included. There are three relaxation times which are proportional to their corresponding transport coefficients, namely:

$$\tau^0 \sim \lambda^0, \qquad \bar{\tau} \sim \bar{\lambda}, \qquad \tau \sim \kappa, \tag{2.12}$$

and two cross-effect coupling coefficients $N$ and $M$. (The proportionality factors in equation (2.12) also may be dependent on $T$ and $\rho$.) Our assumption is based on the following facts.

† Instead of $\pi$ one may use $\pi' = \pi/3$ and $N' = 3N(\mathcal{N}' = 3\mathcal{N})$. This last choice leads to the more symmetrical equations.

(*a*) The equations (2.9)–(2.11) and also (2.12) were deduced by Müller (1967) for an isotropic continuum on the basis of thermodynamical considerations. The same result for a relativistic fluid continuum, covering this classical 'low-temperature' case, was also arrived at recently by Israel (1976).

(*b*) The form of equations (2.9)–(2.12) is in agreement with the corresponding transport equation deduced from kinetic theory by Grad (1949) (equations (5.18) and (5.13) for $\pi$), if second-order terms are dropped.

(*c*) The relaxation terms on the left-hand side also may be derived by application of relaxation theory, or the causality principle, expressing the fact that the dissipative fluxes, i.e. viscous stresses and heat flux, arise solely in the time-retarded response of the driving effect (see § 1 and e.g. Thurston 1964, Kranyš 1966b, 1967).

The argument (*a*) by itself could be considered as sufficient justification of our equations (2.9)–(2.11), and the arguments (*b*) and (*c*), besides providing additional justification for (2.9)–(2.11), show how the results of kinetic theory are almost independent of the kinetic model used, and of approximation methods as well. Let us mention one more argument in favour of our transport equations.

(*d*) It is a very difficult task to deduce directly from microscopic theories the macroscopic equations of some solid, so as to prove our assumption, and even if it could be done for a particular solid there always remains the delicate question of how closely the solid approaches an abstract continuum. Let me propose the following intuitive argument. Thermal energy in a solid is transported mainly by two mechanisms; by quantized electronic excitations, which are called free electrons, and by quanta of lattice vibrations, i.e. phonons. The quanta undergo collisions of a dissipative nature, giving rise to thermal resistance in a medium. Both are usually described as a gas of quasi-particles satisfying the Boltzmann equation, so we have to expect that the appropriate transport equations for these quasi-particles will be very similar to those obtained from the kinetic theory for a gas of molecules. Actually, the relaxation time for heat flux of some solids was estimated by Peierls (1955) (see also Prohofsky and Krumhansl 1964).

The aforementioned constitutive equations have to be completed by conservation laws for mass, linear momentum and energy (i.e. first principle of thermodynamics) in their standard forms, namely:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\ \partial_l v_l=0, \tag{2.13}$$

$$\rho\frac{\mathrm{d}v_l}{\mathrm{d}t}+\partial_k\overset{*}{\tau}_{lk}=0, \qquad \text{or} \qquad \rho\frac{\mathrm{d}v_l}{\mathrm{d}t}+\partial_k\sigma_{lk}+\tfrac{1}{3}\,\partial_l\pi+\partial_k\tau_{lk}=0\ , \tag{2.14†}$$

$$\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}+\overset{*}{\tau}_{lk}\frac{\mathrm{d}e_{lk}}{\mathrm{d}t}+\partial_r q_r=0, \qquad \text{or} \qquad \rho\frac{\mathrm{d}\epsilon}{\mathrm{d}t}+\sigma_{lk}\frac{\mathrm{d}e_{lk}}{\mathrm{d}t}+\frac{1}{3}\pi\frac{\mathrm{d}e_{ss}}{\mathrm{d}t}+\tau_{lk}\frac{\mathrm{d}e_{lk}}{\mathrm{d}t}+\partial_s q_s=0 \tag{2.15}$$

and by the thermodynamical relation which follows:

$$\rho\frac{\mathrm{d}\eta}{\mathrm{d}t}+\partial_l\mathscr{S}_l=\sigma, \qquad \sigma\geqslant 0. \tag{2.16}$$

This is a statement of the entropy balance equation and of the Clausius–Duhem inequality.

† See previous footnote.

In order to show that the transport equations (2.9)–(2.11) are compatible with irreversible thermodynamics we wish to briefly outline the features of Müller's theory. According to Müller one proceeds normally to derive the transport equations which are an approximate system of equations describing a non-equilibrium state of the material continuum not far from thermodynamic equilibrium. This derivation is based on the energy balance equation (2.15)† and the entropy balance equation (2.16) in which we also have to rigorously retain terms of order two, O(2)‡ (and in the resulting transport equations, we obtain only terms up to order one, O(1)). All quantities in the thermodynamic equilibrium state, i.e. the natural state, which do not disappear (e.g. $\rho$, $T$), are considered to be of O(0); we consider all quantities to be of O(1) which vanish at equilibrium (e.g. $q_l$, $\pi$, $\tau_{kl}$) as well as the derivatives of those quantities. The novelty here is not the inclusion of O(2) terms in the entropy balance equation‡ but the inclusion of O(2) terms in Gibbs' equation§ and in the entropy flux definition.

The constitutive assumptions according to Müller may be roughly summarized in the following way. In conventional thermodynamics the internal energy $\epsilon = \epsilon(e_{kl}, T)$ (equation (2.3)) possesses certain important distinguishing characteristics. It is a state function, that is, it is independent of the process followed in changing the state of the body. We may say that the internal energy is a function only of non-dissipative variables which are usually one order lower than dissipative variables. This conclusion is one also accepted by Müller and therefore from this point of view, there is no difference between a reversible $\epsilon'$ and an irreversible $\epsilon$. On the contrary, the quantities appearing in the entropy balance equation (which is the expression for the *fully independent*, intrinsic second principle), i.e. $\eta$ and $\mathscr{S}_l$, are designed to describe the irreversibility of non-equilibrium processes and so they must also be explicitly dependent on the dissipative variables $q_l$, $\pi$ and $\tau_{kl}$. Thus the starting assumption for dependence of specific entropy must be, for example:

$$\eta = \eta(e_{kl}, T, q_l, \pi, \tau_{kl}), \tag{2.17}$$

which is different from $\eta'(e_{kl}, T)$ given by equation (2.3). By differentiation and some thermodynamical considerations, retaining only terms up to O(2), one can deduce the Gibbs' equation:

$$T\rho\frac{\mathrm{d}\eta}{\mathrm{d}t} = \rho\frac{\mathrm{d}\epsilon}{\mathrm{d}t} + \sigma_{kl}\frac{\mathrm{d}e_{kl}}{\mathrm{d}t} - \frac{\tau}{\kappa T}q_l\frac{\mathrm{d}q_l}{\mathrm{d}t} - \frac{1}{3}\frac{\tau^0}{\lambda^0}\pi\frac{\mathrm{d}\pi}{\mathrm{d}t} - \frac{\bar{\tau}}{\bar{\lambda}}\tau_{kl}\frac{\mathrm{d}\tau_{kl}}{\mathrm{d}t}, \tag{2.18}$$

where only the first three terms are usually kept, while the remaining terms, all of O(2), are not. The same considerations that apply to $\eta$ apply also to the entropy flux $\mathscr{S}_l$. Thus, we give a generalized definition of $\mathscr{S}_l$ as:

$$\mathscr{S}_l = \frac{1}{T}(q_l - N\pi q_l - M\tau_{kl}q_k). \tag{2.19}$$

Here, also, the conventional first term on the right-hand side is of O(1); the remaining ones are of O(2).

† The energy balance equation (2.15) as well as the equation of motion (2.14) and the equation of continuity (2.13) can also be simplified later, if desired, by retaining only the terms of order O(1) which correspond to all intents and purposes to the linearization of these equations. This will be done in this paper.

‡ Actually, in conventional theory as will become clear, some second-order terms in the entropy balance equation also appear, e.g. terms originating from $\partial_l(q_l/T)$, though some are missing.

§ Only one such O(2) term $\sigma_{kl}\dot{e}_{kl}$ (if $A = 0$) is considered in conventional theory.

Upon combining equations (2.16), (2.18), (2.19) and (2.15), retaining only the terms up to O(2) and using

$$\tau_{kl}\dot{e}_{kl}=\tau_{kl}\langle\dot{e}_{kl}\rangle, \qquad \tau_{kl}\partial_k q_l=\tau_{kl}\langle\partial_k q_l\rangle, \qquad \tau_{kl}\dot{\tau}_{kl}=\tau_{kl}\langle\dot{\tau}_{kl}\rangle \tag{2.20}$$

and finally also equations (2.9)–(2.11), we obtain the explicit form for entropy production:

$$\sigma=\frac{1}{T}\left(\frac{1}{\kappa T}q_l q_l+\frac{1}{3\lambda^0}\pi^2+\frac{1}{\lambda}\tau_{kl}\tau_{kl}\right)\geqslant 0. \tag{2.21}$$

Conversely, the requirement $\sigma\geqslant 0$ demands the quadratic form in dissipative quantities like equation (2.21) which together with equations (2.18), (2.19) and (2.16), lead to the choice of relations (2.9)–(2.11) and (2.12) which we wanted to show.

It is evident that in $\sigma$, as given by equation (2.21), all terms are of O(2), as they are in conventional parabolic theories. Due, however, to the fact that in equations (2.18) and (2.19) some terms of O(2) are missing, in the usual theories the expressions $\sigma_{\text{conventional}}$ does not contain *all* terms of O(2). Consequently this is reflected in the incompleteness of the transport equations (2.9)–(2.11) where some terms of O(1) are missing. This is the reason why the entropy balance equation and transport equations in conventional theory are incomplete, the consequence of which is a violation of the causality principle.

We also note that once we have derived transport equations (in which all terms of O(1) are present), we no longer need the entropy balance equation. Then this equation, which has a somewhat special role, is necessary only if some further purely thermodynamic considerations are needed.

The first law of thermodynamics (2.15) can be written as follows if use is made of equations (2.4) and (2.5):

$$\rho c_e\frac{\mathrm{d}T}{\mathrm{d}t}+\beta T\frac{\mathrm{d}e_{ss}}{\mathrm{d}t}+\partial_l q_l=0, \tag{2.22}$$

where $\frac{1}{3}\pi\dot{e}_{ss}$ and $\tau_{kl}\dot{e}_{kl}$ have been neglected being terms of O(2), as we have limited ourselves here to the linearized theory (see first footnote on previous page). Using (by virtue of equations (2.5) and (2.6))

$$-\partial_k\sigma_{lk}=\partial_l A+(\lambda+\mu)\partial_l(\partial_k u_k)+\mu\,\partial_k\partial_k u_l-\beta\,\partial_l T \tag{2.23}$$

in equation (2.14), recalling that $v_l=\dot{u}_l$ and setting $A=0$ we arrive at the system of fourteen equations: (2.13), (2.14), (2.22), (2.9), (2.10) and (2.11) for fourteen unknown functions†:

$$\rho, \quad u_l, \quad T, \quad \pi, \quad \tau_{kl}, \quad q_l. \tag{2.24}$$

So the above mentioned system of equations, which is of 14th+3rd order (as equation (2.14) is of 2nd order in $u_l$) determines the unknown functions without recourse to the notions involved in the second principle. The equations (2.16) and (2.18) may be used, for example, when one wishes for some reason to know $\sigma$ or $\eta$ or some thermodynamical function derived from $\eta$, such as the free energy $\psi=\epsilon-\eta T$ etc. The proof that the proposed theory is really hyperbolic is given in § 6.

The resulting form of stress–strain dependence for our dissipative elastic solid follows from equations (2.1), (2.5), (2.7), (2.9) and (2.10):

† The quantities (2.24) will be called simply 'moments' as in a kinetic theory. Therefore a fourteen-moment discription means fourteen unknowns or a fourteen equations description.

$$\sigma_{lk}+\mathscr{T}_{lk}+(\tau^0-\bar{\tau})\tfrac{1}{3}\delta_{lk}\dot{\mathscr{T}}_{ss}+\bar{\tau}\dot{\mathscr{T}}_{lk}$$
$$=-\{[A-\beta(T-T_0)]\delta_{lk}+(\lambda\delta_{lk}e_{ss}+2\mu e_{lk})+(\lambda'\delta_{lk}\dot{e}_{ss}+2\mu'\dot{e}_{lk})$$
$$+[(3\lambda^0 N-\bar{\lambda}M)\tfrac{1}{3}\delta_{lk}\partial_s g_s+\bar{\lambda}\partial_{(l}g_{k)}]\}. \tag{2.25}$$

By contraction $l=k$:

$$\sigma_{ss}+\pi+\tau^0\dot{\pi}=-[\![\{3[A+\lambda e_{ss}-(T-T_0)\beta]+2\mu e_{ss}\}+\lambda^0\dot{e}_{ss}+3\lambda^0 N\,\partial_s q_s]\!] \tag{2.26}$$

and we come back to equations (2.5) and (2.9). Equation (2.10) can be verified easily.

Equation (2.25) represents our generalized form of the Voigt (or Maxwell–Voigt) constitutive equation, or, in other words, equation (2.25) is 'Hooke's law' generalized to a dissipative elastic medium.

## 3. The linearized fourteen-moment equations and their Fourier transforms

We limit ourselves to the case of an unbounded space filled with an immobile isotropic dissipative elastic medium in thermodynamic equilibrium and at the mechanical equilibrium reference state of no tension ($A=0$) in which there is a *forced disturbance* of very small amplitude. Therefore, we assume that the governing equations which are quasilinear may be linearized near this equilibrium reference state, so that all coefficients of differential equations will be considered as having constant values corresponding to the reference state (i.e. in the coefficients we set $T\to T_{eq}=T_0$, $\rho\to\rho_{eq}$ and we will drop the suffix eq).

The fourteen equations governing our problem are equations (2.13), (2.14), (2.22), (2.9), (2.10) and (2.11) which, after being rearranged (as is mentioned in the text before equation (2.24)) and linearized read:

$$\dot{\rho}+\rho\,\partial_s\dot{u}_s=0, \qquad \left(\text{now only } \dot{\rho}=\frac{\partial\rho}{\partial t}\right) \tag{3.1}$$

$$\rho\ddot{u}_l-(\lambda+\mu)\partial_l(\partial_k u_k)-\mu\,\partial_k\partial_k u_l+\beta\,\partial_l T+\tfrac{1}{3}\partial_l\pi+\partial_k\tau_{lk}=0, \tag{3.2}$$

$$\rho c_e\dot{T}+\beta T\partial_s\dot{u}_s+\partial_s q_s=0, \tag{3.3}$$

$$\pi+\tau^0\dot{\pi}+\lambda^0(\partial_s\dot{u}_s+3N\partial_s q_s)=0, \tag{3.4}$$

$$\tau_{lk}+\bar{\tau}\dot{\tau}_{lk}+\bar{\lambda}(\langle\partial_l\dot{u}_k\rangle+M\langle\partial_l q_k\rangle)=0, \tag{3.5}$$

$$q_l+\tau\dot{q}_l+\kappa(\partial_l T+NT\,\partial_l\pi+MT\,\partial_s\tau_{ls})=0. \tag{3.6}$$

### 3.1. *Solution by Fourier transform*

In seeking a solution to the system of fourteen linear partial differential equations (3.1)–(3.6), we assume each of the unknown functions (2.19) to have the form of a propagating *plane wave*:

$$Q-Q_{eq}=\hat{Q}\,e^{i(\omega t-(\boldsymbol{k}\cdot\boldsymbol{x}))} \qquad (\boldsymbol{k}=(0,0,K)) \tag{3.7}$$

where we have chosen the wavevector $\boldsymbol{k}$ orientated along the $x_3$ axis. ($\omega$ is a real number and $K$ is complex.)

The Fourier transform of the linearized set of equations (3.1)–(3.6) after some rearrangement, may be put in the matrix form (3.8):

$$
\left(\begin{array}{cccc|cccc|cccccc}
-\rho\omega^2+\mu K^2 & \cdot & -\mathrm{i}K & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \omega\bar{B} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\frac{\mathrm{i}}{2}\frac{\bar{\lambda}}{\bar{\tau}}\omega K & \cdot & \omega\bar{B} & -\frac{1}{2}\frac{\bar{\lambda}}{\bar{\tau}}MK & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -\frac{\kappa}{\tau}MTK & \omega\bar{Z} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & -\rho\omega^2+\mu K^2 & \cdot & -\mathrm{i}K & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \omega\bar{B} & \cdot & \cdot & \cdot & \frac{\mathrm{i}}{3}\frac{\bar{\lambda}}{\bar{\tau}}\omega K & \cdot & \cdot & \cdot & \frac{1}{3}\frac{\bar{\lambda}}{\bar{\tau}}MK \\
\cdot & \cdot & \cdot & \cdot & -\frac{\mathrm{i}}{2}\frac{\bar{\lambda}}{\bar{\tau}}\omega K & \cdot & \omega\bar{B} & -\frac{1}{2}\frac{\bar{\lambda}}{\bar{\tau}}MK & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{\kappa}{\tau}MTK & \omega\bar{Z} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \omega & -\mathrm{i}\rho\omega K & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\rho\omega^2+(\lambda+2\mu)K^2 & -\mathrm{i}\beta K & -\mathrm{i}K & -\frac{\mathrm{i}}{3}K & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mathrm{i}\beta T\omega K & \omega\rho c_e & \cdot & \cdot & -K \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{2}{3}\mathrm{i}\frac{\bar{\lambda}}{\bar{\tau}}\omega K & \cdot & \omega\bar{B} & \cdot & -\frac{2}{3}\frac{\bar{\lambda}}{\bar{\tau}}MK \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mathrm{i}\frac{\lambda^0}{\tau^0}\omega K & \cdot & \cdot & \omega\bar{N} & -3\frac{\lambda^0}{\tau^0}NK \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{\kappa}{\tau}K & -\frac{\kappa}{\tau}MTK & -\frac{\kappa}{\tau}NTK & \omega\bar{Z}
\end{array}\right)
\left(\begin{array}{c}
\hat{u}_1 \\ \hat{\tau}_{12} \\ \hat{\tau}_{13} \\ \hat{q}_1 \\ \hline \hat{u}_2 \\ \hat{\tau}_{22} \\ \hat{\tau}_{23} \\ \hat{q}_2 \\ \hline \hat{\rho} \\ \hat{u}_3 \\ \hat{T} \\ \hat{\tau}_{33} \\ \hat{\pi} \\ \hat{q}_3
\end{array}\right) = 0 \qquad (3.8)
$$

where

$$\bar{N} = 1 - \frac{\mathrm{i}}{\tau^0 \omega}, \qquad \bar{B} = 1 - \frac{\mathrm{i}}{\bar{\tau}\omega}, \qquad \bar{Z} = 1 - \frac{\mathrm{i}}{\tau\omega}. \tag{3.9}$$

In the system of equations (3.8) $\hat{\tau}_{11}$ is no longer present due to the relation $\tau_{ss} = 0$ (equation (2.4)), as we must work only with independent unknowns. Because three equations are of second order in equation (3.8), our problem is really of order 17.

The algebraic system of homogeneous equations (3.8) have a non-trivial solution if, and only if, the appropriate determinant of this system vanishes, namely:

$$\Delta_{17}(W, \omega) = 0, \qquad (W = \omega/K). \tag{3.10}$$

This is the characteristic equation, and its solutions (eigenvalues $W = W(\omega)$) define the dispersion dependence of the complete set of eigenmodes belonging to our system.

As is evident from equation (3.8) $\Delta_{17}$ is equal to the product of three lower order determinants: $\Delta_{17} = \Delta_5 \Delta_5 \Delta_7$. Hence instead of the dispersion equation (3.10) we need investigate only the two much simpler equations:

$$\Delta_5 = 0, \tag{3.11}$$

$$\Delta_7 = 0. \tag{3.12}$$

Evidently equation (3.11) corresponds to waves with transverse polarization directed along the axes $x_1$ or $x_2$, while equation (3.12) corresponds to a wave with longitudinal polarization, i.e. polarization directed along $x_3$.

## 4. Transverse waves

### 4.1. *General case (fourteen- or thirteen-moment description)*

The possible phase velocities $W = \omega/K$, with a transverse polarization, are given using equations (3.11) and (3.8) in dimensionless form by the equation:

$$\Delta_5 = \text{constant} \times \overset{\perp}{W}\bar{B} \begin{vmatrix} 1 - \overset{\perp}{W}{}^2 & -\mathrm{i} & \\ -\tfrac{1}{2}\mathrm{i}\bar{\Lambda}_\perp \overset{\perp}{W} & \bar{B}\overset{\perp}{W} & -\tfrac{1}{2}\bar{\Lambda}_\perp \overset{\perp}{\mathcal{M}} \dfrac{c_\perp}{v} \\ & -\Lambda_\perp \overset{\perp}{\mathcal{M}} \dfrac{n}{c_\perp} & \bar{Z}\overset{\perp}{W} \end{vmatrix} = 0 \tag{4.1}$$

where

$$c_\perp^2 = \frac{\mu}{\rho}, \qquad \overset{\perp}{W} = \frac{W}{c_\perp}, \qquad \bar{\Lambda}_\perp = \frac{\bar{\lambda}}{\bar{\tau}} \frac{1}{\rho c_\perp^2}, \qquad \Lambda_\perp = \frac{\kappa}{\tau} \frac{1}{\rho c_e c_\perp^2}, \qquad \overset{\perp}{\mathcal{M}} = \rho M c_\perp v, \qquad v^2 = c_e T. \tag{4.2}$$

This equation can be reduced to the form

$$\Delta_5 \equiv \text{constant} \times \overset{\perp}{W}(\overset{\perp}{W}{}^2 - \overset{\perp}{W}{}^2_{\mathrm{I}})(\overset{\perp}{W}{}^2 - \overset{\perp}{W}{}^2_{\mathrm{II}}) = 0 \tag{4.3}$$

where

$$\overset{\perp}{W}{}^2_{\mathrm{I,II}} = \frac{1}{2\tilde{A}}[\tilde{B} \pm (\tilde{B}^2 - 4\tilde{A}\tilde{C})^{1/2}] \tag{4.4}$$

and

$$\tilde{A} = \bar{B}\bar{Z}, \qquad \tilde{B} = \bar{B}\bar{Z} + \tfrac{1}{2}\bar{\Lambda}_\perp \bar{Z} + \tfrac{1}{2}\bar{\Lambda}_\perp \Lambda_\perp \overset{\perp}{\mathcal{M}}{}^2, \qquad \tilde{C} = \tfrac{1}{2}\bar{\Lambda}_\perp \Lambda_\perp \overset{\perp}{\mathcal{M}}{}^2. \tag{4.5}$$

Therefore we can write down equation (4.4) also as

$$2\overset{\perp}{W}{}^2_{\mathrm{I,II}} = \left[\left(1 + \frac{1}{2}\bar{\Lambda}_\perp \frac{1}{\bar{B}}\right) + \frac{\tilde{C}}{\tilde{A}}\right] \pm \left\{\left[\left(1 + \frac{1}{2}\bar{\Lambda}_\perp \frac{1}{\bar{B}}\right) - \frac{\tilde{C}}{\tilde{A}}\right]^2 + 2\bar{\Lambda}_\perp \frac{1}{\bar{B}}\frac{\tilde{C}}{\tilde{A}}\right\}^{1/2}. \tag{4.6}$$

The complete characteristic polynomial for possible transverse waves, i.e. in both polarization directions $x_1$ and $x_2$, is

$$(\Delta_5)^2 = 0, \tag{4.7}$$

$\Delta_5$ being given by equation (4.3). (Equation (4.3) also says that for each polarization direction the propagation equation has the differential operator (called hyperbolic if $W_{\mathrm{I,II}} < \infty$):

$$\partial_t(\partial_t^2 - W_{\mathrm{I}}^2(\omega)\Delta)(\partial_t^2 - W_{\mathrm{II}}^2(\omega)\Delta) \qquad (\Delta = \partial_s\partial_s). \tag{4.8}$$

In this expression the derivatives of lower order than fifth have been converted in the frequency dependence of $W = W(\omega)$.)

The zero root of equation (4.3) has to be associated with the mass flow velocity, which was chosen in our case to be zero. Then we have two modes† for transverse waves. Because $\overset{\perp}{W}{}^2_{\mathrm{I}} \geqslant \overset{\perp}{W}{}^2_{\mathrm{II}}$, we will call the I-wave a fast transverse (or quasi-mechanical) wave and the II-wave a slow transverse (or quasi-thermal) wave, whose existence is mainly due to heat conduction (as will be shown at the end of this section).

The complex phase velocities $W_{\mathrm{I,II}}$, depending on the wave frequency through the expressions $\bar{B}(\omega)$ and $\bar{Z}(\omega)$ (see equation (3.9)), give us information on both the effective phase speed $W^+ = \omega/\mathrm{Re}\, K$ and the coefficient of absorption $\omega/\mathrm{Im}\, K$.

*4.1.1. Limiting case when $\omega \to \infty$.* The wavefront speed (signal speed) $V$ for each wave mode can be found either from the complex $W$ or from the real phase velocity $W^+$ as a limit:

$$V = \lim_{\omega\to\infty} W(\omega) = \lim_{\omega\to\infty} W(\bar{B}(\omega), \bar{Z}(\omega), \bar{N}(\omega)) \tag{4.9}$$

but $\bar{B}(\infty) = \bar{Z}(\infty) = \bar{N}(\infty) = 1$ by (2.9).

*4.1.2. Limiting case when $\kappa$, $\bar{\lambda}$, $\lambda^0 \to 0$.* If the thermal conductivity coefficient decreases to zero $\kappa \to 0$, which also means $\tau \to 0$ by equation (2.12), then because of equation (2.11) $q_l \to 0$, meaning elimination of heat conduction (and also of equation (2.11)) from the description. The same applies to $\bar{\lambda}$ and $\lambda^0$, so

$$\text{if} \begin{cases} \kappa \to 0 \\ \bar{\lambda} \to 0 \\ \lambda^0 \to 0 \end{cases} \text{ or equivalently } \begin{cases} |\bar{Z}| \to \infty \\ |\bar{B}| \to \infty \\ |\bar{N}| \to \infty \end{cases} \text{ then } \begin{cases} q_l \to 0 \\ \tau_{ls} \to 0 \\ \pi \to 0. \end{cases} \tag{4.10}$$

† We call $W^2$ a mode, which means one wave propagating in the positive $(+W)$ and one in the negative $(-W)$ sense, with the same speeds.

(It is quite natural to require $\bar{\lambda} \neq 0$ and $\kappa = 0$, but the requirement $\bar{\lambda} = 0$ and $\kappa \neq 0$ although formally acceptable, is not so natural in the kinetic theory where $\bar{\lambda}$ and $\kappa$ are connected together† if the higher order moment, i.e. heat flux, is retained. Accordingly we will call the second possibility a heuristic case.)

*4.1.3. Limiting case when $\omega \to 0$.*

$$\lim_{\omega \to 0} \bar{Z} = \lim_{\omega \to 0} \left(1 - \frac{\mathrm{i}}{\tau\omega}\right) \to 1 - \mathrm{i}\infty, \tag{4.11}$$

or equivalently

$$|\bar{Z}| \to \infty, \qquad |\bar{B}| \to \infty, \qquad |\bar{N}| \to \infty \tag{4.12}$$

but these results are those for an adiabatic dissipation-free case, because the simultaneous transitions

$$\tau \to 0, \qquad \bar{\tau} \to 0, \qquad \tau^0 \to 0, \tag{4.13}$$

also lead to the conditions (4.12), which, according to equation (4.10) lead to a cancellation of all dissipative fluxes and therefore to the adiabatic state.

### *4.2. The adiabatic case (five-moment description)*

This case is included as a special, dissipation-free case of wave propagation when $\omega \to 0$ or by equation (4.12), when $|\bar{Z}| \to \infty$, and $|\bar{B}| \to \infty$ ($|\bar{N}| \to \infty$ need not be considered as transverse waves are independent of $\bar{N}$). Then from equations (4.5) and (4.6) it follows that

$$\overset{\perp}{W}{}_{\mathrm{I}}^2 = \frac{\tilde{B}}{\tilde{A}} = 1, \qquad \overset{\perp}{W}{}_{\mathrm{II}}^2 = 0, \qquad \left(\frac{\tilde{C}}{\tilde{A}} = 0\right) \tag{4.14}$$

which is by virtue of definition (4.2) a well known result.

### *4.3. The case with shear viscosity only (ten-moment description)*

This case with no heat conduction follows from equations (4.5)–(4.6) when we allow $|\bar{Z}| \to \infty$ (see equation (4.10)). Doing this, we obtain:

$$\overset{\perp}{W}{}_{\mathrm{I}}^2 = \frac{\tilde{B}}{\tilde{A}} = 1 + \frac{1}{2}\bar{\Lambda}_{\perp}\frac{1}{\bar{B}(\omega)}, \qquad \overset{\perp}{W}{}_{\mathrm{II}}^2 = 0, \qquad \left(\frac{\tilde{C}}{\tilde{A}} = 0\right) \tag{4.15‡}$$

which means that the effective phase speed $W^+$ is (writing $\Omega = \bar{\tau}\omega$)

$$|\overset{\perp}{W}{}_{\mathrm{I}}^+| = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{\Omega^2}\right)^{1/2}\left\{\left[\left(1 + \frac{1}{2}\bar{\Lambda}_{\perp}\right) + \frac{1}{\Omega^2}\right]^2 + \frac{\bar{\Lambda}_{\perp}^2}{4\Omega^2}\right\}^{-1/2}$$
$$\times \left[\!\left[\left\{\left[\left(1 + \frac{1}{2}\bar{\Lambda}_{\perp}\right) + \frac{1}{\Omega^2}\right]^2 + \frac{\bar{\Lambda}_{\perp}^2}{4\Omega^2}\right\}^{1/2} + \left[\left(1 + \frac{1}{2}\bar{\Lambda}_{\perp}\right) + \frac{1}{\Omega^2}\right]\right]\!\right]^{1/2}. \tag{4.16}$$

† To give some idea of the orders of magnitude of the newly introduced coefficients, their values for a monatomic gas (see Grad 1949) where the bulk viscosity is not included are:

$$\frac{\tau}{\bar{\tau}} = \frac{3}{2}, \quad \bar{\lambda} = 2p\bar{\tau}, \quad \kappa = \frac{5}{2}pR\tau, \quad \frac{\kappa}{\bar{\lambda}} = \frac{15}{8}R, \quad M = \frac{2}{5p} \quad \text{and} \quad N = 0.$$

‡ This result is obtained also if we set the coupling coefficient shear/heat $\overset{\perp}{\mathcal{M}} = 0$ in equations (4.5) and (4.6).

The wavefront speed equation (4.9) of the fast mode I (equation (4.15)) is

$$\overset{\perp}{V}_{\mathrm{I}}=|\overset{\perp}{W}_{\mathrm{I}}(\infty)|=\left(1+\frac{1}{2}\bar{\Lambda}_{\perp}\right)^{1/2} \quad \text{or} \quad V_{\mathrm{I}}=\left(\frac{\mu}{\rho}+\frac{1}{2\rho}\frac{\bar{\lambda}}{\bar{\tau}}\right)^{1/2} \quad (\bar{\lambda}=2\mu'), \tag{4.17}$$

a value greater than that in the adiabatic case and this value is very sensitive to the shear viscosity relaxation time which, when set to zero (as is usual in conventional parabolic theories), results in infinite signal speed. It is evident from equation (4.16) that for any frequency $\omega>0$, $\overset{\perp}{W}^{+}(\omega)>\overset{\perp}{W}^{+}_{\text{adiabatic}}$.

### 4.4. *The case without viscosity* (*eight-moment description*)

This case with heat conduction follows if we allow $|\bar{B}|\to\infty$, but then the result is the same as in the adiabatic case. We may conclude that the shear viscosity is mainly responsible for the dispersion of transverse waves.

We have already seen in § 4.3 that the absence of heat flow results in a stopping of the slow II-wave; and that in § 4.4 the absence of shear viscosity results in a disappearance of dispersion and identification of the I-wave with the adiabatic transverse wave. Therefore the II-wave could also be called a transverse quasi-thermal wave. This mode is predicted here for the first time whereas the mode I (see equation (4.4)), which had already been treated (see Thurston 1964, equation (389)), is generalized due to the heat/viscosity cross effect.

## 5. Longitudinal waves

### 5.1. *General case* (*fourteen-moment description*)

The possible phase velocities $W=\omega/K$ with longitudinal polarization are given by the following equation, using equations (3.12) and (3.8) (in dimensionless form):

$$\Delta_7\equiv\text{constant}\times\overset{\mathrm{L}}{W}\begin{vmatrix} 1-\overset{\mathrm{L}}{W}^2 & -\mathrm{i}\beta' & -\mathrm{i} & -\frac{\mathrm{i}}{3} & \cdot \\ -\mathrm{i}p'\overset{\mathrm{L}}{W} & \overset{\mathrm{L}}{W} & \cdot & \cdot & -1 \\ -\mathrm{i}\frac{2}{3}\bar{\Lambda}\overset{\mathrm{L}}{W} & \cdot & \bar{B}\overset{\mathrm{L}}{W} & \cdot & -\frac{2}{3}\bar{\Lambda}\mathcal{M}\frac{c_{\mathrm{L}}}{v} \\ -\mathrm{i}\Lambda^0\overset{\mathrm{L}}{W} & \cdot & \cdot & \bar{N}\overset{\mathrm{L}}{W} & -3\Lambda^0\mathcal{N}\frac{c_{\mathrm{L}}}{v} \\ \cdot & -\Lambda & -\Lambda\mathcal{M}\frac{v}{c_{\mathrm{L}}} & -\Lambda\mathcal{N}\frac{v}{c_{\mathrm{L}}} & \bar{Z}\overset{\mathrm{L}}{W} \end{vmatrix}=0 \tag{5.1}$$

where

$$c_{\mathrm{L}}^2=\frac{\lambda+2\mu}{\rho} \qquad \overset{\mathrm{L}}{W}=\frac{W}{c_{\mathrm{L}}}, \qquad p'=\frac{\beta T}{\rho c_{\mathrm{L}}^2}, \qquad \beta'=\frac{\beta}{\rho c_e}, \qquad v^2=c_eT, \qquad \mathcal{M}=\rho Mc_{\mathrm{L}}v,$$

$$\delta=p'\beta', \qquad \mathcal{N}=\rho Nc_{\mathrm{L}}v, \qquad \bar{\Lambda}=\frac{\bar{\lambda}}{\bar{\tau}}\frac{1}{\rho c_{\mathrm{L}}^2}, \qquad \Lambda^0=\frac{\lambda^0}{\tau^0}\frac{1}{\rho c_{\mathrm{L}}^2}, \qquad \Lambda=\frac{\kappa}{\tau}\frac{1}{\rho c_e c_{\mathrm{L}}^2}. \tag{5.2}$$

This equation can be reduced to the form

$$\Delta_7 \equiv \text{constant} \times \overset{\mathrm{L}}{W}{}^3(\overset{\mathrm{L}}{W}{}^2 - \overset{\mathrm{L}}{W}{}^2_{\mathrm{I}})(\overset{\mathrm{L}}{W}{}^2 - \overset{\mathrm{L}}{W}{}^2_{\mathrm{II}}) = 0, \tag{5.3}$$

where

$$\overset{\mathrm{L}}{W}{}^2_{\mathrm{I,II}} = \frac{1}{2\tilde{A}}[\tilde{B} \pm (\tilde{B}^2 - 4\tilde{A}\tilde{C})^{1/2}], \tag{5.4}$$

and

$$\tilde{A} = \bar{B}\bar{N}\bar{Z}, \tag{5.5}$$

$$\tilde{B} = (1+\delta)\bar{B}\bar{N}\bar{Z} + \tfrac{2}{3}(\bar{\Lambda}\bar{N} + \tfrac{1}{2}\Lambda^0\bar{B})\bar{Z} + \Lambda\bar{B}\bar{N} + \tfrac{2}{3}\Lambda\bar{\Lambda}\mathcal{M}^2\bar{N} + \tfrac{1}{3}\Lambda\Lambda^0(3\mathcal{N})^2\bar{B}, \tag{5.6}$$

$$\tilde{C} = \Lambda\bar{B}\bar{N} + \frac{2}{3}\Lambda\bar{\Lambda}\left(1 + (1+\delta)\mathcal{M}^2 - 2\beta'\frac{v}{c_{\mathrm{L}}}\mathcal{M}\right)\bar{N}$$
$$+ \frac{1}{3}\Lambda\Lambda^0\left(1 + 3(1+\delta)(3\mathcal{N})^2 - 2\beta'\frac{v}{c_{\mathrm{L}}}(3\mathcal{N})\right)\bar{B} + \frac{2}{9}\Lambda\Lambda^0\bar{\Lambda}(\mathcal{M} - 3\mathcal{N})^2. \tag{5.7}$$

From equation (5.3) we see that there are once again two longitudinal modes, and because of $\overset{\mathrm{L}}{W}{}^2_{\mathrm{I}} \geqslant \overset{\mathrm{L}}{W}{}^2_{\mathrm{II}}$ (from equation (5.4)), we call the I-wave a fast longitudinal wave or 'quasi-mechanical wave' (or sound wave) and the II-wave a slow wave or 'quasi-thermal wave'. Let us turn to some special cases.

### 5.2. *The adiabatic case (five-moment description)*

This case can be obtained from equation (4.12) as a limiting case for $|\bar{B}| \to \infty$, $|\bar{N}| \to \infty$ and $|\bar{Z}| \to \infty$. Effecting those limits on equations (5.4)–(5.7), we obtain

$$\overset{\mathrm{L}}{W}{}^2_{\mathrm{I}} = 1+\delta, \qquad \overset{\mathrm{L}}{W}{}^2_{\mathrm{II}} = 0, \qquad \left(\text{as } \frac{\tilde{C}}{\tilde{A}} = 0\right) \tag{5.8}$$

where $\delta$ is the so called thermoelastic (dimensionless) coupling coefficient:

$$\delta \equiv p'\beta' = \frac{\beta^2 T}{\rho^2 c_e c_{\mathrm{L}}^2} = \frac{\beta^2 T^2}{\rho^2 v^2 c_{\mathrm{L}}^2} \tag{5.9}$$

(by equation (5.2)). The well known result $\overset{\mathrm{L}}{W}{}^2_{\mathrm{I}} = 1$ follows from equation (5.8) for the 'uncoupled' case, i.e. when stress (2.5) is not directly influenced by heating ($\beta = 0$; $\delta = 0$).

### 5.3. *The case when dissipation is due only to viscosity (eleven-moment description)*

This case follows from the general formulae (5.4)–(5.7) with $|\bar{Z}| \to \infty$ (i.e. elimination of heat flux; see equation (4.10)) which lead to the expressions

$$\overset{\mathrm{L}}{W}{}^2_{\mathrm{I}} = (1+\delta) + \frac{2}{3}\bar{\Lambda}\frac{1}{\bar{B}(\omega)} + \frac{1}{3}\Lambda^0\frac{1}{\bar{N}(\omega)}; \qquad \overset{\mathrm{L}}{W}{}^2_{\mathrm{II}} = 0, \tag{5.10}$$

telling us that the slow quasi-thermal mode disappears and the main acoustical mode survives giving, as a result of bulk and shear viscosity, a higher phase velocity than the adiabatic sound speed. This would probably be a satisfactory approximation for a lot of practical purposes.

If we eliminate either bulk viscosity ($|\bar{N}| \to \infty$) (resulting in a thirteen-moment description) or shear viscosity ($|\bar{B}| \to \infty$) (leading to a nine-moment description) both propagation modes survive; only the dispersion curves are modified accordingly.

### 5.4. *The heuristic case when dissipation is due to heat conduction only (eight-moment description)*

This case follows from equations (5.4)–(5.7) with $|\bar{B}| \to \infty$ and $|\bar{N}| \to \infty$ leading to the two non-trivial modes:

$$2\overset{L}{W}{}^2_{\mathrm{I,II}} = \left((1+\delta)+\frac{\Lambda}{\bar{Z}}\right) \pm \left[\left((1+\delta)-\frac{\Lambda}{\bar{Z}}\right)^2 + 4\delta\frac{\Lambda}{\bar{Z}}\right]^{1/2}, \tag{5.11}$$

as was found by Popov (1967) (equation (7)) and others. This same case was studied also by Chadwick and Sneddon (1958) who used the Fourier transport equation. A consequence of this is that the governing equations are not completely hyperbolic. Anyway, the heuristic cases† cannot be trusted much due to neglect of viscosity effects which are of the same order as the heat conduction effect, which has been retained.

## 6. The hyperbolicity of the theory

The requirement that our system of fourteen partial differential equations (3.1)–(3.6) be hyperbolic can be formulated (Courant and Hilbert 1966, §§ III, 3 and 6) in the following way. If the characteristic equation of the system under consideration, which in our case is the characteristic polynomial $\Delta_{17} = (\Delta_5)^2\Delta_7$ ($\Delta_5$ and $\Delta_7$ being given by equations (4.3) and (5.3) respectively) in the limit $\omega \to \infty$:

$$\lim_{\omega\to\infty} \Delta_{17} \equiv \text{constant} \times \lim_{\omega\to\infty} (\overset{\perp}{W}{}^2 - \overset{\perp}{W}{}^2_{\mathrm{I}})^2(\overset{\perp}{W}{}^2 - \overset{\perp}{W}{}^2_{\mathrm{II}})^2\overset{\perp}{W}{}^2(\overset{L}{W}{}^2 - \overset{L}{W}{}^2_{\mathrm{I}})(\overset{L}{W}{}^2 - \overset{L}{W}{}^2_{\mathrm{II}})\overset{L}{W}{}^3 = 0 \tag{6.1}$$

possesses *only real and finite solutions for all roots*, then the system is hyperbolic. First, five zero roots $W^5 = 0$, as well as $\overset{\perp}{W}{}^2_{\mathrm{I,II}}$ (given by equation (4.4)) and $\overset{L}{W}{}^2_{\mathrm{I,II}}$ (given by equation (5.4)), fulfill this condition of reality and finiteness as long as $\bar{\Lambda} < \infty$, $\Lambda^0 < \infty$ and $\Lambda < \infty$. This is fulfilled if and only if, simultaneously (cf equation (5.2)), we have

$$\bar{\tau} > 0, \qquad \tau^0 > 0, \qquad \tau > 0, \tag{6.2}$$

because all our coefficients in (4.4) and (5.4) are real.

If only one of the relaxation constants is equal to zero then all the propagation wave modes have infinite front speeds. We notice that relaxation constants (6.2) are *definitely responsible* for the finiteness of wavefront speeds and therefore for guaranteeing the *causality principle* and *therefore cannot be neglected.*

## 7. Comparison with other theories and results

This will help us determine the differences and weaknesses of those particular theories. Usually, either: (i) the heat flux is introduced via a stationary Fourier law (see § 7.4), then the whole system of equations is not hyperbolic; or (ii) one introduces the relaxed

† See § 4.1.2, last paragraph.

heat flux equation but the internal viscosity is omitted (see § 7.3); or (iii) heat flux is neglected completely (see §§ 7.1 and 7.2). Let us consider first the possibility (iii) where the main weakness is the almost complete disregarding of thermodynamics (equations (2.15) and (2.11) have been dropped) and transport equations for viscosity are usually reduced to one (corresponding to equation (2.9)) as only one-dimensional propagation is being studied.

### 7.1. *The four-moment description*

This is the most simple description without the use of a transport equation, but where only an equation of motion is used (as the continuity equation (2.13) need not be considered if we are not interested in density variations), where the dissipation is taken into account simply by the inclusion of the frictional damping force which is proportional to the velocity; namely,

$$\rho\frac{\mathrm{d}v_l}{\mathrm{d}t}+\rho b v_l+\partial_k\sigma_{lk}=0, \qquad v_l=\frac{\mathrm{d}u_l}{\mathrm{d}t}. \tag{7.1}$$

This equation with equation (2.23) (where $\beta=0$) leads to the telegraph equation for both transverse and longitudinal waves:

$$\ddot{u}_l+b\dot{u}_l=c^2\,\Delta u_l, \qquad (c=c_\perp, c_\mathrm{L}) \tag{7.2}$$

which is hyperbolic. The phase speed is

$$W^+(\omega)=c\sqrt{2}\left[1+\left(1+\frac{b^2}{\omega^2}\right)^{1/2}\right]^{-1/2}\leqslant c \qquad (W^+(\infty)=c) \tag{7.3}$$

showing that the dispersion dependence goes in the opposite sense compared to our results where $W^+(\omega)>W^+_{\text{adiabatic}}$ which has been confirmed by experiments both in fluids and solids (see § 8). References to equation (7.2) can be found mostly only in older literature (cf Lamb 1925 § 23); nevertheless equation (7.2) was used also by Weber (1961, equation (8.34)), whose work was of course criticized (see e.g. Maugin 1974).

### 7.2. *The six-moment description*

(No shear viscosity or heat flux is considered.) This was applied to the propagation of longitudinal waves by Hillier (1949) (letting aside $\rho$, this is a third-order problem) along a viscoelastic filament where a stress–strain relation, which is the combination of Maxwell (1890) and Meyer–Kelvin–Voigt (Voigt 1892) formulae, was used (see Kolsky 1953, equation (5.44)) or a formally identical expression is that of Voigt with relaxation (see Thurston 1964, equation (381)):

$$\sigma+\frac{\eta_v}{E'_a+E_v}\dot{\sigma}=\frac{E_vE'_a}{E'_a+E_v}\epsilon+\frac{\eta_vE'_a}{E'_a+E_v}\dot{\epsilon}. \tag{7.4}$$

In this way, Hillier deduced an hyperbolic third-order equation for quite reasonable dispersion and absorption curves. It is clear that (7.4) is of considerable formal similarity with our equation (2.26). However, there are two important differences. First, equation (2.26) includes a coupling to heat flux via the coefficient $N$. In equation (7.4), of course, heat is not considered but, even if it were, one is obliged to use equation (7.4) in the same form, because from Maxwell–Voigt theory there are no indications on how to include such a generalization. Second, in equation (2.26) only the irreversible part of $\overset{*}{\sigma}_{lk}$, i.e. $\mathcal{T}_{lk}$, is relaxed (see term $\tau^0\dot{\pi}$), as required by time irreversibility of dissipation processes†, while in equation (7.4) $\sigma$ plays the role of our total stress tensor $\overset{*}{\sigma}_{lk}$. This is an important difference as for a Maxwell solid model, and models derived from it, the *total* stress is irreversible. The same applies to Boltzmann's generalization (often thought to be always sufficiently general) of the stress–strain relation. Firstly, coupling with heat conduction is totally missing; secondly, if the memory function is an odd (as it usually is) function of time, total stress on the left-hand side is, of course, irreversible.

The Voigt stress–strain relation alone is not correct for high frequencies as it leads to the parabolic 'wave equation' whereas Maxwell's relation alone, resulting in an hyperbolic wave equation, has another inconvenience: if a solid is strained by a definite amount and held at this strain, the stress will relax with time. The Hillier theoretical dispersion and absorption curves are, at least qualitatively, confirmed by experiments (see Kolsky 1953, p 161, Nolle 1949, Ivey *et al* 1949).

### 7.3. *An eight-moment description*

This description, neglecting both shear and bulk viscosity but using a *relaxed* heat flux equation like equation (2.11) (with $N = M = 0$ of course) was proposed and used by many authors (e.g. Popov 1967, Lord and Schulman 1967, Achenbach 1968, Norwood and Warren 1969). Their results conform to phase speed dispersion for the corresponding special case outlined in § 5.3. Two longitudinal modes are given by equation (5.11). Of course no dissipation of transverse waves exists according to this description.

### 7.4. *The conventional fourteen-moment description*

This may be called a description using Voigt's stress–strain relation as a viscosity transport equation (equation (2.25) with $\tau^0 = \bar{\tau} = N = M = 0$) and Fourier's law (equation (2.11) with $\tau = N = M = 0$) as a heat flow transport equation. Taking into account that:

$$\lim_{\bar{\tau}\to 0}\frac{\bar{\Lambda}}{\bar{\bar{B}}} = \lim_{\bar{\tau}\to 0}\frac{\bar{\lambda}}{\rho c^2}\frac{1}{[\bar{\tau}-(\mathrm{i}/\omega)]} = \mathrm{i}\omega\frac{\bar{\lambda}}{\rho c^2} = \mathrm{i}\omega(\bar{\Lambda}\bar{\tau}); \qquad \lim_{\tau^0\to 0}\frac{\Lambda^0}{\bar{\bar{N}}} = \mathrm{i}\omega\frac{\lambda^0}{\rho c^2}, \qquad \lim_{\tau\to 0}\frac{\Lambda}{\bar{\bar{Z}}} = \mathrm{i}\omega\frac{\kappa}{c_e\rho c^2} \tag{7.5}$$

and $N = M = 0$ (i.e. $\mathcal{N} = \mathcal{M} = 0$) in formulae (4.5), (4.6) and then in equations (5.4)–(5.7), we obtain:

$$\overset{+}{W}{}^2_{\mathrm{I}} = 1 + \mathrm{i}\omega\frac{1}{2}\frac{\bar{\lambda}}{\rho c_\perp^2} = 1 + \tfrac{1}{2}\mathrm{i}\omega(\bar{\Lambda}_\perp\bar{\tau}), \qquad \overset{+}{W}{}^2_{\mathrm{II}} = 0, \tag{7.6}$$

for the transverse wave modes, and

$$\frac{\tilde{B}}{\bar{\bar{A}}} = (1+\delta) + \mathrm{i}\omega[\tfrac{4}{3}(\bar{\Lambda}\bar{\tau}) + \tfrac{1}{3}(\Lambda^0\tau^0) + (\Lambda\tau)], \qquad \frac{\tilde{C}}{\bar{\bar{A}}} = \mathrm{i}\omega(\Lambda\tau)\{1 + \mathrm{i}\omega[\tfrac{4}{3}(\bar{\Lambda}\bar{\tau}) + \tfrac{1}{3}(\Lambda^0\tau^0)]\}, \tag{7.7}$$

for the longitudinal modes. This description is valid only for nearly stationary processes, and the results (7.6) and (7.7) only for the relatively low frequency range

† See § 1.

$\sup\{\bar{\tau}\omega, \tau^0\omega, \tau\omega\} \ll 1$. The front speeds of all propagating modes are infinite (see equations (4.9), (7.6) and (7.7)), as is the case in every parabolic theory. Often, authors applying such descriptions to wave propagation discuss only wave absorption in order to avoid discussion of the absurd dispersion dependence for high frequencies. This has been a practice also in fluid theory since the time of Kirchhoff's (1868) publications.

We have seen that the proposed non-stationary fourteen-moment theory is therefore more general than any particular example given above and that it covers every typical case (commonly used at the present time) except the case outlined in § 7.1. We see further that it unifies them, moreover giving finite front speeds for all propagation modes. The proposed theory requires the knowledge of five new coefficients, of which those relaxation times, occasionally used, may be estimated more easily than $N$ and $M$ (for some materials $\tau \sim 10^{-11}$–$10^{-13}$ s; see Peierls 1955, p 135; $\tau^0$ and $\bar{\tau}$ probably only differ by a few orders of magnitude†). If one sets, for simplicity, $N = M = 0$, then dynamical couplings between dissipation fluxes are removed and the formulae (5.4) and (4.4) are simplified as a result and, furthermore, the slow transverse wave disappears ($\overset{\perp}{W}_{\mathrm{II}} \to 0$)‡.

## 8. Experimental investigation of wave propagation in dissipative continua and solids

There exists a great amount of literature on this subject, and we can therefore limit ourselves to some selected sources. As a general reference, let us quote Mason (1966–75). Let us at least try to choose some experimental results connected with the results of the analysis made in this paper, which are the dispersion and adsorption frequency dependences of existing propagation modes in unbounded dissipative solids.

Theoretically it was found that the phase velocity at high frequency of each mode is greater than at low frequency and signal speed (at $\omega \to \infty$) is finite. A plot of $\overset{\mathrm{L}}{W}{}_{\mathrm{I}}^{+}$ or $\overset{\perp}{W}{}_{\mathrm{I}}^{+}$ against $\ln(\omega\tau)$ has the form of a typical '$S$' dispersion step with an inflection point near $\omega\tau = 1$, whereas $\overset{\mathrm{L}}{W}{}_{\mathrm{II}}^{+}$ or $\overset{\perp}{W}{}_{\mathrm{II}}^{+}$ rises almost linearly turning near $\omega\tau = 1$ and then approaches asymptotically its signal value. The absorption curve for I-waves when plotted against $\ln(\omega\tau)$ is a bell-shaped curve with a maximum near $\omega\tau = 1$.

The general dispersion feature was verified by observation in monatomic and polyatomic gases (see Greenspan 1965, Mitin and Yakovlev 1971, Kneser 1965), at least for the longitudinal fast mode. Gases are covered of course explicitly by Müller's theory§, which in the case of monatomic gases, coincides formally with the linearized thirteen-moment method of Grad. Müller's theory also covers fluids for which some

† $\tau^0$ and $\bar{\tau}$ might be estimated based on Maxwell–Voigt stress–strain relations.

‡ The constitutive equations of microviscoelasticity (Eringen 1967b, equations (3.18) and (3.19)) in the relaxed form (but without dynamical coupling) compatible with the present thermodynamics are as follows:

$$\pi + \tau^{(1)}\dot{\pi} = (3\lambda' + 2\mu' + \kappa')\dot{\epsilon}_{ss}, \qquad s + \tau^{(2)}\dot{s} = (3\alpha' + \beta' + \gamma')\dot{\phi}_{r,r},$$

$$\tau_{kl} + \tau^{(3)}\dot{\tau}_{kl} = (2\mu' + \kappa')\langle\dot{\epsilon}_{kl}\rangle + \kappa'\dot{\epsilon}_{[kl]}, \qquad s_{kl} + \tau^{(4)}s_{kl} = (\beta' + \gamma')\langle\dot{\phi}_{k,l}\rangle + (\beta' - \gamma')\dot{\phi}_{[k,l]},$$

where

$$\tau_{kl} \equiv {}_{D}t_{kl} - \tfrac{1}{3}\delta_{kl\,D}t_{ss}, \qquad s_{kl} \equiv {}_{D}m_{kl} - \tfrac{1}{3}\delta_{kl}m_{ss}, \qquad \tau \equiv {}_{D}t_{ss}, \qquad s \equiv {}_{D}m_{ss}.$$

Of course, the reversible parts of the $t_{kl}$ and $m_{kl}$ remain non-relaxed, namely:

$${}_{E}t_{kl} = \lambda\epsilon_{rr}\delta_{kl} + (\mu + \kappa)\epsilon_{kl} + \mu\epsilon_{kl}, \qquad {}_{E}m_{kl} = \alpha\phi_{r,r}\delta_{kl} + \beta\phi_{k,l} + \gamma\phi_{l,k}.$$

§ The formula (36) in Müller (1967) is incorrect. The correct value of heat signal speed is given in Kranyš (1972), equation (5.52).

experimental results can be found in Litovitz and Davis (1965). For the measurement of dispersion and absorption in solids we refer to Kolsky (1953, chap 6, and references therein), particularly to Nolle (1949), Ivey *et al* (1949) and to Knopoff (1965). Sometimes it is difficult to find out 'impartial' measurements of the dispersion, based uniquely on observation, as the experimentalists try to interpret their results, or calculate some necessary auxiliary qualities using conventional (often suspect) formulae especially if they are using the indirect method also based on some theory.

## 9. Conclusions

A phenomenological theory for a dissipative elastic solid whose equations form a hyperbolic system is proposed. The Müller–Israel non-stationary transport equations for dissipative fluxes containing new cross-effect terms, as required for compatability with the entropy principle expressed by the balance equation with all second-order terms, have been adopted in order to guarantee physical causality and the possibility of describing fast, transient processes. In the adaptation of Müller's theory, which is a continuum theory, to elasticity, the principal step was the replacement of the viscosity tensor $p_{ij}$ by the purely irreversible (or dissipative) part of the stress tensor $\mathcal{T}_{ij}$ in the transport equations. The theory formed from the system of fourteen partial-differential equations, of total order seventeen, is hyperbolic. Five new transport coefficients (in general dependent on temperature and density), appear in the transport equations, in contrast to conventional parabolic theories: however, three of them (relaxation times) have been investigated previously†, in connection with some simpler constitutive equations like the Maxwell–Voigt laws. It would be desirable to have experimental values of those constants necessary to make a quantitative comparison between theory and experiments for various solids‡.

The complete system of propagation modes has been determined from the fourteen linearized equations. There are four mutually distinct non-trivial propagation modes; two for longitudinal waves, and two for transverse waves. The slow transverse mode is predicted here for the first time, while improved expressions for the remaining modes have been found. The wavefront of each mode propagates with a finite velocity, proving directly the hyperbolicity of the theory. The wavefront speeds of the modes (being always higher than those for dissipation-free propagation), represent the speed of propagation of the characteristic surfaces on which a discontinuity of some quantities can occur, and therefore represent weak shock wave speeds, or, more exactly, shock precursor speeds.

Many results on wave propagation, based on various parabolic theories of different order (but never higher than order 17, which is our case), have been published. The parabolic theories of course lead to an infinite signal speed for almost all propagation modes! Such theories are thus applicable only to phenomena which can be called 'quasi-stationary', i.e. slowly varying on space and time scales characterized by $W_{\mathrm{I}}\tau$ and the relaxation time $\tau$. This is not satisfied for many phenomena involving steep gradients or rapid variations, an example of which would be just a fast change near the front of the wave pulse. For conventional theories containing relaxation terms in

† More exactly two relaxation coefficients appearing in the Maxwell and Maxwell–Voigt relations play a similar but not necessarily identical role to our $\tau$ and $\bar{\tau}$. See § 1.

‡ See first footnote on p 700.

transport equations like those of a Maxwell solid and Voigt–Maxwell solid respectively, thermodynamics has not been constructed†.

It is reasonable to expect that the front speed of a thermal (or diffusion‡ or viscosity) wave *is finite* and well defined, its value being of the order of the speed of adiabatic sound. In the past this has not been accepted. A diffusion-like parabolic equation $\dot{u} = \alpha \, \Delta u$ was conventionally used for the description of those phenomena and it was believed that there was no definite velocity which could be defined as the velocity of heat or diffusion. The argument for this was, for example, given by Maxwell. If we attempt to measure this velocity by measuring the time necessary for the production of a given amount of disturbance at a given distance from the source of the disturbance, we find that the smaller the selected value of the disturbance the greater the velocity will appear to be, for however great the distance, and however small the time, the value of the disturbance will differ mathematically from zero. Even now it is difficult, if not impossible, to measure directly, with a prescribed accuracy, the heat (or diffusion) signal speed. Fortunately it is possible to do it indirectly by the measurement of the speed of high frequency harmonic components of sound accompanied by heat conduction. These speeds converge gradually to the definite signal speed.

It was Maxwell (1867) himself who derived, from kinetic theory, a non-stationary transport equation for heat, but he then treated the time derivative of heat flux as negligible in most cases and thus confirmed the old Fourier law. The next occasion where that old inadequacy of transport theory could be definitively removed was missed, when the so called Hilbert–Chapman-Enskog normal solution method was applied (1912–22) to solve the Boltzmann kinetic equation. As this method (in principle, a small parameter method requiring $\omega\tau \ll 1$§), contains assumptions which result in the elimination of temporal derivative terms of transport equations, the form of the Fourier (and Fock) law was confirmed, which greatly increased the confidence of physicists in atemporal transport laws.

Recent development in kinetic theory (Grad 1949‖) have confirmed the almost simultaneous, great number of propositions for the improvement of the phenomenological constitutive laws for dissipative fluxes made by many authors (see Cattaneo 1948, 1958, Vernotte 1958, 1961, Eringen 1960, Nettleton 1960, Chester 1963, Prohofsky and Krumhansl 1964, Kaliski 1965, Kranyš 1966b, Weymann 1967, Popov 1967, Lord and Schulman 1967, McCarthy 1970a, b, Simons 1972, Kranyš and Teichmann 1974, and others). These have, consequently, been placed on a more sound theoretical basis. Nevertheless, parabolic theories are still popular enough to occupy their unjustified place in the textbooks of physics.

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† See § 1.

‡ The derivation of the hyperbolic equation for diffusion based on a random walk assumption was given e.g. by Fock (1926) and is repeated by Weymann (1967). Diffusion signal speeds in monatomic gases or plasmas were studied by Kranyš and Teichmann (1974).

§ Many authors have a tendency to overprice the 'nice' agreement of Chapman–Enskog solutions with experiments around $\omega\tau \approx 1$ where this approach is no longer valid!

‖ See the first footnote of the paper.

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